

Winter School 2020

Why do we care about rotating flows?

- Earth rotates!

Obviously want to understand flows relevant for weather & climate

Also differently!

Why?

Two amazing facts

- (1) Rapidly rotating flows are "quasi two dimensional"
- (2) In 2D turbulent flows, energy builds up at large scales

What are the implications of this?

- 1) Rotating flows are characterized by the emergence of large-scale structures \rightarrow e.g., jets
- 2) Rotating turbulence can produce long-lived coherent structures
 \rightarrow 3D turbulence is inherently disequilibrium process
- 3) Friction plays a very different role in 2D and 3D turbulence

When is rotation important?

How do we know whether rotation is important for a given flow?

When do we need to consider the Earth's rotation

- tornado
- midlatitude cyclone
- bathubs

How fast does Nairn's tank need to rotate before we need to take into account its motion?

Need to understand effects of rotation quantitatively

The momentum equation:

Applicable to the atmosphere (and seen by many of you)
(see General circulation notes online for "derivation")

Navier-Stokes equation:

$$\frac{D\mathbf{u}}{Dt} = \underbrace{\frac{g}{\rho}}_{\text{inertia}} - \underbrace{\frac{1}{\rho} \nabla p}_{\text{gravitational force}} + \underbrace{2\nabla^2 \mathbf{u}}_{\text{frictional force}}$$

pressure gradient force

Navier-Stokes represents Newton's Second Law, as such is valid in an inertial reference frame.

\Rightarrow Rotating Earth is not an inertial reference frame
results in extra "forces" in momentum equations

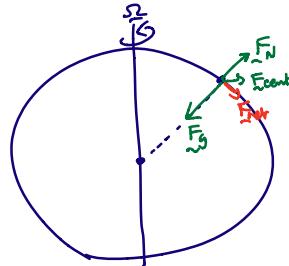
$$\text{Centrifugal force} \quad F_{cent} = \underline{\omega} \times \underline{\omega} \times \underline{v}$$

$$\text{Coriolis force} \quad F_{cor} = 2\underline{\omega} \times \underline{u}$$

Centrifugal Force

- acts on all objects, stationary or moving
- Directed away from Earth's rotation axis

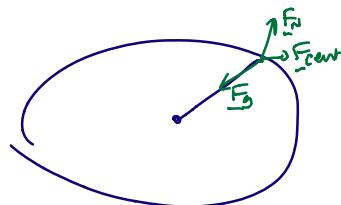
Consider a stationary object on a spherical planet:



Since $F_g \parallel F_N$, cannot balance F_{cent} : net force is non-zero!

Why are objects on Earth continually pushed sideways?

Earth is not spherical, but is an oblate spheroid



\Rightarrow "Down" as measured by a plumb bob is not towards centre of the Earth

\Rightarrow Makes sense to combine Centrifugal & gravitational force (via "geopotential")

\Rightarrow The combined gravitational-centrifugal force points "down", so can treat just like gravity

Coriolis force

⇒ acts on objects moving in the rotating frame

⇒ acts at right angles to the direction of motion

⇒ does no work.

Coriolis force has profound implications for the flow, as we shall see.

Tangent plane equations

Useful form of the equations (with cartesian geometry)

$$\frac{Du}{Dt} = fu - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad \text{where } f = 2\omega \sin \phi$$

Can derive these equations (see Vallee)

↑
angular rotation frequency

When are the rotational terms (fu , fv) important?

Non-dimensionalisation of u-momentum equation

Assume a length scale of relevance L

and a typical velocity scale v

then have an advection timescale $T = \frac{L}{v}$

Define non-dimensionalised variables $\hat{u} = \frac{u}{v}$, $\hat{t} = \frac{t}{T} = \frac{tv}{L}$, $\hat{x} = \frac{x}{L}$, $\hat{y} = \frac{y}{L}$, $\hat{z} = \frac{z}{H}$

where H is the depth scale of the flow, and $H/L \ll 1$

Define non-dimensionalised variables (with "hats")

$$\begin{aligned} \hat{u} &= \hat{u}U & t &= \frac{\hat{t}L}{U} & x &= \hat{x}L \\ \hat{v} &= \hat{v}V & & & y &= \hat{y}L \end{aligned}$$

And $\hat{z} = \frac{z}{H}$ where H is the depth scale

$$\text{and } \frac{H}{L} \ll 1$$

Substitute the above equations into the momentum equation

$$\frac{Du}{Dt} = f\hat{v} - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{\partial^2 \hat{u}}{\partial z^2} \right)$$

$$\left(\frac{U}{L}\right) \frac{D\hat{u}}{Dt} = f\hat{v}U - \frac{1}{\rho L} \frac{\partial p}{\partial x} + \nu \left(\frac{U}{L^2} \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{U}{L^2} \frac{\partial^2 \hat{u}}{\partial y^2} + \frac{U}{H^2} \frac{\partial^2 \hat{u}}{\partial z^2} \right)$$

$$\Rightarrow \left(\frac{U^2}{L}\right) \frac{D\hat{u}}{Dt} = Uf\hat{v} - \frac{1}{\rho L} \frac{\partial p}{\partial x} + \frac{\nu U}{L^2} \left(\cancel{\frac{\partial^2 \hat{u}}{\partial x^2}} + \cancel{\frac{\partial^2 \hat{u}}{\partial y^2}} + \frac{L^2}{H^2} \frac{\partial^2 \hat{u}}{\partial z^2} \right)$$

Now, since we assume $L \gg H$, vertical component of Laplacian dominates

Dividing by $\frac{U^2}{L}$ we have

$$\frac{D\hat{u}}{Dt} = \left(\frac{fL}{U}\right) \hat{v} - \frac{1}{\rho U^2} \frac{\partial p}{\partial x} + \left(\frac{\nu}{UL}\right) \left(\frac{L}{H}\right)^2 \frac{\partial^2 \hat{u}}{\partial z^2}$$

We have the non dimensional parameters

$$\text{Rossby number } Ro = \frac{U}{fL}$$

$$\text{Reynolds number } Re = \frac{UL}{\nu}$$

$$\text{Aspect ratio } A = \frac{H}{L}$$

$$\frac{D\hat{u}}{Dt} = \frac{\hat{u}}{Ro} - \frac{1}{\rho U^2} \frac{\partial p}{\partial z} + \frac{1}{Re A^2} \frac{\partial^2 \hat{u}}{\partial z^2}$$

For $Re A \gg 1 \Rightarrow$ friction unimportant compared to inertia

For $Ro \sim 0(1)$ or greater \Rightarrow rotation is important

What are the dominant terms under different conditions?

$$v = 1 \times 10^{-5} \text{ m}^2/\text{s}^2, \quad f = 1 \times 10^{-4} \text{ s}^{-1}$$

[midlatitude cyclone]

$$H = 10 \times 10^3 \text{ m} \quad \Rightarrow \quad Re A^2 = 1 \times 10^8$$

$$L = 1000 \times 10^3 \text{ m} \quad Ro = 0.1$$

$$U = 10 \text{ m s}^{-1}$$

\Rightarrow viscosity unimportant, rotation dominant

[bathtub]

$$H = 1 \text{ m} \quad Re A^2 = 1 \times 10^5$$

$$L = 1 \text{ m} \quad \Rightarrow \quad Ro = 4000$$

$$U = 0.1 \text{ m s}^{-1}$$

- Rotation not important
- friction not important

$$\frac{D\hat{u}}{Dt} = \left[\frac{\hat{g}}{R_0} - \frac{1}{\rho U^2} \frac{\partial p}{\partial x} \right] + \frac{1}{ReA} \frac{\partial^2 \hat{u}}{\partial z^2}$$

At low Rossby number, dominant balance is between Coriolis force & pressure gradient force
This is geostrophic balance.

→ Friction only important at very small length scales

$$Re = \frac{UL}{\nu} = \frac{C}{\nu}$$